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Non-local continuity equations and binary Darboux transformation of noncommutative (anti) self-dual Yang–Mills equations

U Saleem, M Hassan and M Siddiq[1]

Department of Physics, University of the Punjab, Quaid-e-Azam Campus, Lahore 54590, Pakistan

E-mail: usman-physics@yahoo.com, mhassan@physics.pu.edu.pk and mohsin-pu@yahoo.com

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Abstract
We present an infinite number of non-local continuity equations of noncommutative (anti) self-dual Yang–Mills (nc-(A)SDYM) equations using the induction method of Brézin *et al* (1979 *Phys. Lett.* B **82** 442) and relate it to the Lax pair and the parametric Bäcklund transformation of the system. From the Lax pair, we derive a binary Darboux transformation to generate solutions of the nc-(A)SDYM equations.

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1. Introduction

A well-known example of a multi-dimensional integrable system is the (anti) self-dual Yang–Mills ((A)SDYM) theory [1–14]. The (A)SDYM equations admit integrability structures such as the existence of the Lax pair, the Bäcklund transformation and the existence of an infinite number of continuity equations [1–14]. The four-dimensional (A)SDYM equations reduce to two-dimensional integrable equations such as the sine/sinh-Gordon equation, Liouville equation, KdV equation and principal chiral field equation [9–11]. Recently, there has been an increasing interest in the noncommutative field theories where a field theory is deformed in the sense of Moyal $\star$-product deformation[2] [15]. It has been shown that, in general,

[1] On study leave from PRD (PINSTECH) Islamabad, Pakistan.
[2] The noncommutativity of space coordinates is defined as

$$[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu},$$

where $\theta^{\mu\nu}$ is a second rank antisymmetric tensor. The product of two functions in noncommutative space is defined as

$$(f \star g)(x) \equiv \exp\left(\tfrac{\mathrm{i}}{2}\left(\theta^{\mu\nu}\partial_\mu^{x_1}\partial_\nu^{x_2}\right)\right) f(x_1)g(x_2)|_{x_1=x_2=x} = f(x)g(x) + \frac{\mathrm{i}\theta^{\mu\nu}}{2}\partial_\mu f(x)\partial_\nu g(x) + \vartheta(\theta^2),$$

where $\partial_\mu^{x_i} = \frac{\partial}{\partial x_i^\mu}$ and $i = 1, 2$. The $\star$-product is associative $(f \star g) \star h = f \star (g \star h)$.

noncommutativity of time variables leads to non-unitarity and affects the causality of the theory [16, 17]. The noncommutative extension of integrable equations has also attracted a great deal of interest during the last decade and these noncommutative versions of integrable equations have been shown to admit integrability structures such as the Lax pair, the Bäcklund transformation, the existence of an infinite number of conserved quantities and zero-curvature representation [18–28].

The noncommutative extension of the (A)SDYM equations has been investigated by different authors (e.g. [22–25] and references therein). The nc-(A)SDYM equations reduce to the ordinary (A)SDYM equations in the limit when the deformation parameter θ vanishes. Many two-dimensional noncommutative integrable equations such as noncommutative versions of Korteweg de Vries (KdV) equation, nonlinear Schrödinger equation, principal chiral field equations, etc are shown to be obtained by dimensional reduction of nc-(A)SDYM equations (see [24, 25] and references therein). The ADHM (Atiyah–Drinfeld–Hitchin–Manin) construction of the noncommutative instanton solutions of nc-(A)SDYM equation has led to some important consequences about the integrability of the system and has been further extended to twistor interpretation of the ADHM construction and multi-instanton solutions of nc-(A)SDYM (e.g. [29–35]). A great deal of work has been done concerning noncommutative solitons in gauge theory and string theory (see, e.g., [36–47]). In the context of string theory these solutions play very important role and describe noncommutative branes (e.g. [48, 49]). Particularly, these models and their reductions correspond to D-brane configurations of D0–D4 brane in $N = 2$ string theory with a B-field background that induces on D-branes a noncommutative generalization of a modified $U(n)$ sigma model in $(2 + 1)$ dimensions [50–53]. Furthermore, the dressing method has been employed for constructing multisoliton solutions of noncommutative modified $U(n)$ sigma model [54]. Since all the reduced systems of nc-(A)SDYM equations exhibit integrability properties such as linearization, Lax pair, existence of an infinite number of continuity equations, multisoliton solutions, Darboux transformation, etc, therefore, it becomes natural to investigate the existence of an infinite number of continuity equations and Darboux transformation for nc-(A)SDYM equations. Due to the matrix nature of the objects involved, the generalization to noncommutative case works in the same way as in the commutative case. But from the point of view of the integrability of nc-(A)SDYM equations, it is worthwhile to study the existence of an infinite number of conservation laws of the system. In this paper, we show that the method of Brézin *et al* [55] generalizes to the case of nc-(A)SDYM equations so that the system exhibits an infinite sequence of conservation laws. The other aspect that we have investigated for our system is the solution generating method in the form of a binary Darboux transformation. The hope is that this might explain geometrical structure of noncommutative twistor approach and moduli space dynamics in the context of nc-(A)SDYM and particularly dynamics of noncommutative D-branes in string theory. The existence of an infinite number of non-local continuity equations and the binary Darboux transformation of nc-(A)SDYM equations can also give some insight into understanding certain algebraic structures and non-perturbative properties of $N = 2$ string field theory.

The paper is organized as follows. In section 2, we give a review of the noncommutative generalization of the (A)SDYM equations and write its Lax pair. In section 3, we use the Lax pair to obtain an infinite number of continuity equation (conservation laws) of the system using induction and present the Bäcklund transformation for the conservation laws following the commutative construction developed in [7]. In section 4, we develop a binary Darboux transformation for generating solutions of the system and compare some results with those obtained by dressing method.

2. nc-(A)SDYM equations—an overview

Let us start by defining Yang–Mills fields on four-dimensional noncommutative Euclidean space E^4. The curvature of 2-form is defined as

$$F^{\star}_{\mu\nu} = \partial\mu A^{\star}_{\nu} - \partial\nu A^{\star}_{\mu} - [A^{\star}_{\mu}, A^{\star}_{\nu}]_{\star}, \tag{1}$$

where $A^{\star}_{\mu}$ are the Yang–Mills fields being $n \times n$ matrix-valued 1-forms representing $U(n)$[3] connections and $[A^{\star}_{\mu}, A^{\star}_{\nu}]_{\star}$ is commutator of the fields in noncommutative space defined as $[A^{\star}_{\mu}, A^{\star}_{\nu}]_{\star} = A^{\star}_{\mu} \star A^{\star}_{\nu} - A^{\star}_{\nu} \star A^{\star}_{\mu}$.

The coordinates $x_\mu, \mu = 0, 1, 2, 3$, on noncommutative Euclidean space E^4 are related to the coordinates on noncommutative complex Euclidean space as

$$x_y = x_0 + \mathrm{i}x_3, \qquad x_{\bar{y}} = x_0 - \mathrm{i}x_3,$$
$$x_z = x_1 + \mathrm{i}x_2, \qquad x_{\bar{z}} = x_1 - \mathrm{i}x_2.$$

The metric becomes $\mathrm{d}s^2 = \mathrm{d}x_y\,\mathrm{d}x_{\bar{y}} + \mathrm{d}x_z\,\mathrm{d}x_{\bar{z}}$ and the Yang–Mills fields ($U(n)$ connections) are expressed as

$$A^{\star}_{y} = A^{\star}_{0} + \mathrm{i}A^{\star}_{3}, \qquad A^{\star}_{\bar{y}} = A^{\star}_{0} - \mathrm{i}A^{\star}_{3},$$
$$A^{\star}_{z} = A^{\star}_{1} + \mathrm{i}A^{\star}_{2}, \qquad A^{\star}_{\bar{z}} = A^{\star}_{1} - \mathrm{i}A^{\star}_{2}.$$

The nc-(A)SDYM equations now expressed in complex coordinates are written as

$$F^{\star}_{yz} \equiv \partial_y A^{\star}_{z} - \partial_z A^{\star}_{y} - [A^{\star}_{y}, A^{\star}_{z}]_{\star} = 0, \tag{2}$$

$$F^{\star}_{\bar{y}\bar{z}} \equiv \partial_{\bar{y}} A^{\star}_{\bar{z}} - \partial_{\bar{z}} A^{\star}_{\bar{y}} - [A^{\star}_{\bar{y}}, A^{\star}_{\bar{z}}]_{\star} = 0, \tag{3}$$

$$F^{\star}_{y\bar{y}} + F^{\star}_{z\bar{z}} = 0. \tag{4}$$

Equations (2) and (3) imply that the connection components can be expressed as

$$A^{\star}_{y} = g^{-1} \star \partial_y g, \qquad A^{\star}_{\bar{y}} = \bar{g}^{-1} \star \partial_{\bar{y}} \bar{g}, \tag{5}$$

$$A^{\star}_{z} = g^{-1} \star \partial_z g, \qquad A^{\star}_{\bar{z}} = \bar{g}^{-1} \star \partial_{\bar{z}} \bar{g}, \tag{6}$$

where $g, \bar{g}$ and their inverses with respect to $\star$-product $g^{-1}, \bar{g}^{-1}$ are functions of $y, \bar{y}, z, \bar{z}$ and belong to Lie group $U(n)$. The nc-(A)SDYM equations (2)–(4) can also be obtained as the compatibility condition of the following linear system:

$$(\partial_y + \lambda\partial_{\bar{z}})\Psi'(y, \bar{y}, z, \bar{z}; \lambda) = (A^{\star}_{y} + \lambda A^{\star}_{\bar{z}}) \star \Psi'(y, \bar{y}, z, \bar{z}; \lambda), \tag{7}$$

$$(\partial_z - \lambda\partial_{\bar{y}})\Psi'(y, \bar{y}, z, \bar{z}; \lambda) = (A^{\star}_{z} - \lambda A^{\star}_{\bar{y}}) \star \Psi'(y, \bar{y}, z, \bar{z}; \lambda), \tag{8}$$

where $\Psi'(y, \bar{y}, z, \bar{z}; \lambda)$ is some $n \times n$ matrix-valued field and λ is the spectral parameter. The compatibility condition of the linear equations (7) and (8) is the flatness condition of the 1-form connection ($\star$-zero-curvature condition)

$$(\partial_z - \lambda\partial_{\bar{y}})(A^{\star}_{y} + \lambda A^{\star}_{\bar{z}}) - (\partial_y + \lambda\partial_{\bar{z}})(A^{\star}_{z} - \lambda A^{\star}_{\bar{y}}) + [A^{\star}_{y} + \lambda A^{\star}_{\bar{z}}, A^{\star}_{z} - \lambda A^{\star}_{\bar{y}}]_{\star} = 0. \tag{9}$$

Equating the coefficients of different powers of λ in the above equation we obtain equations (2)–(4). Now we define Lax operators for the nc-(A)SDYM equations

$$L = D_y + \lambda D_{\bar{z}}, \qquad M = D_z - \lambda D_{\bar{y}},$$

[3] The $SU(n)$ gauge group is not closed under the Moyal commutator because of the noncommutativity of matrices breaks the cyclic property of traces; we, therefore, restrict our analysis to gauge group $U(n)$ that is closed under Moyal commutator.

where the operators D_y, $D_{\bar{y}}$, D_z and $D_{\bar{z}}$ are given by

$$D_y = \partial_y - A_y^\star, \qquad D_{\bar{y}} = \partial_{\bar{y}} - A_{\bar{y}}^\star \qquad D_z = \partial_z - A_z^\star, \qquad D_{\bar{z}} = \partial_{\bar{z}} - A_{\bar{z}}^\star,$$

with

$$[D_y, D_z]_\star = 0, \qquad [D_{\bar{y}}, D_{\bar{z}}]_\star = 0, \qquad [D_y, D_{\bar{y}}]_\star + [D_z, D_{\bar{z}}]_\star = 0.$$

In terms of the matrices L and M, the nc-(A)SDYM equations appear as the compatibility condition of the linear system (7), (8), expressed as

$$L \star \Psi'(y, \bar{y}, z, \bar{z}; \lambda) \equiv (D_y + \lambda D_{\bar{z}}) \star \Psi'(y, \bar{y}, z, \bar{z}; \lambda) = 0,$$
$$M \star \Psi'(y, \bar{y}, z, \bar{z}; \lambda) \equiv (D_z - \lambda D_{\bar{y}}) \star \Psi'(y, \bar{y}, z, \bar{z}; \lambda) = 0,$$

and the zero-curvature condition (9) is equivalent to $[L, M]_\star = 0$. The reality condition satisfied by $\Psi'(y, \bar{y}, z, \bar{z}; \lambda)$ is

$$\Psi'(y, \bar{y}, z, \bar{z}; \lambda) \star [\Psi'(y, \bar{y}, z, \bar{z}; \bar{\lambda})]^\dagger = 1. \tag{10}$$

Now we consider a gauge transformation

$$\Psi(y, \bar{y}, z, \bar{z}; \lambda) = \bar{g} \star \Psi'(y, \bar{y}, z, \bar{z}; \lambda), \tag{11}$$

such that the fields $A_{\bar{y}}^\star$ and $A_{\bar{z}}^\star$ vanish. The gauge fields $A_{\bar{y}}^\star$ and $A_{\bar{z}}^\star$ transform as

$$\begin{aligned} A_{\bar{y}}^\star &\to A_{\bar{y}}'^\star = \bar{g} \star A_{\bar{y}}^\star \star \bar{g}^{-1} + \bar{g} \star \partial_{\bar{y}} \bar{g}^{-1} = 0, \\ A_{\bar{z}}^\star &\to A_{\bar{z}}'^\star = \bar{g} \star A_{\bar{z}}^\star \star \bar{g}^{-1} + \bar{g} \star \partial_{\bar{z}} \bar{g}^{-1} = 0, \end{aligned} \tag{12}$$

where $\bar{g}^{-1}$ is the inverse of $\bar{g}$ with respect to the $\star$-product. The rest of components are

$$A_y'^\star \equiv \mathcal{J}_y^\star = J^{\star -1} \star \partial_y J^\star, \qquad A_z'^\star \equiv \mathcal{J}_z^\star = J^{\star -1} \star \partial_z J^\star,$$

where $J^{\star -1} = \bar{g}^{-1} \star g$ and is the inverse of $J^\star$ with respect to $\star$-product. The gauge-fixed linear system is

$$(\partial_y + \lambda \partial_{\bar{z}}) \Psi(y, \bar{y}, z, \bar{z}; \lambda) = \mathcal{J}_y^\star \star \Psi(y, \bar{y}, z, \bar{z}; \lambda), \tag{13}$$

$$(\partial_z - \lambda \partial_{\bar{y}}) \Psi(y, \bar{y}, z, \bar{z}; \lambda) = \mathcal{J}_z^\star \star \Psi(y, \bar{y}, z, \bar{z}; \lambda). \tag{14}$$

The self-duality equation (4) becomes

$$\partial_{\bar{y}} \mathcal{J}_y^\star + \partial_{\bar{z}} \mathcal{J}_z^\star = 0, \tag{15}$$

which is a continuity equation or a conservation law. The reality condition (10) is invariant under gauge transformation (11), i.e.

$$\Psi(y, \bar{y}, z, \bar{z}; \lambda) \star [\Psi(y, \bar{y}, z, \bar{z}; \bar{\lambda})]^\dagger = 1. \tag{16}$$

Similarly we can have

$$\partial_y \mathcal{J}_{\bar{y}}^\star + \partial_z \mathcal{J}_{\bar{z}}^\star = 0, \tag{17}$$

where

$$\mathcal{J}_{\bar{y}}^\star = \partial_{\bar{y}} J^\star \star J^{\star -1}, \qquad \mathcal{J}_{\bar{z}}^\star = \partial_{\bar{z}} J^\star \star J^{\star -1}.$$

The compatibility condition of the linear equations (13) and (14) is

$$(\partial_z \mathcal{J}_y^\star - \partial_y \mathcal{J}_z^\star + [\mathcal{J}_y^\star, \mathcal{J}_y^\star]_\star) - \lambda(\partial_{\bar{y}} \mathcal{J}_y^\star + \partial_{\bar{z}} \mathcal{J}_z^\star) = 0.$$

The nc-(A)SDYM equation (15) can also be reduced to a noncommutative two-dimensional principal chiral model if we take $x_y = x_{\bar{y}} = x_0$ and $x_z = x_{\bar{z}} = x_1$

$$\partial_0 \mathcal{J}_0^\star + \partial_1 \mathcal{J}_1^\star = 0, \tag{18}$$

where

$$\mathcal{J}_0^\star = J^{\star -1} \star \partial_0 J^\star, \qquad \mathcal{J}_1^\star = J^{\star -1} \star \partial_1 J^\star. \tag{19}$$

The associated linear equations (13) and (14) become

$$(\partial_0 + \lambda \partial_1)\Psi = \mathcal{J}_0^\star \star \Psi, \tag{20}$$

$$(\partial_1 - \lambda \partial_0)\Psi = \mathcal{J}_1^\star \star \Psi. \tag{21}$$

The Lax pairs (13), (14) and (20), (21) generate infinitely many non-local continuity equations which we address in the next section.

3. Non-local continuity equations

In order to derive infinitely many non-local continuity equations (conservation laws) for the nc-(A)SDYM theory, we adopt the induction procedure of Brézin *et al* [55] developed for two-dimensional sigma model and later used by Prasad *et al* [7] for (commutative) (A)SDYM equations. We proceed by defining covariant derivatives in the noncommutative space acting on some scalar function χ, such that

$$\mathcal{D}_y \chi = \partial_y \chi - \mathcal{J}_y^\star \star \chi, \qquad \mathcal{D}_z \chi = \partial_z \chi - \mathcal{J}_z^\star \star \chi,$$

where

$$[\mathcal{D}_y, \mathcal{D}_z]_\star = 0.$$

We now suppose that there exist kth currents $\mathcal{J}_y^{\star(k)}$ and $\mathcal{J}_z^{\star(k)}$ for $k = 1, 2, \ldots, n$ which are conserved, i.e.

$$\partial_{\bar{y}} \mathcal{J}_y^{\star(k)} + \partial_{\bar{z}} \mathcal{J}_z^{\star(k)} = 0, \tag{22}$$

such that

$$\mathcal{J}_y^{\star(k)} = -\partial_{\bar{z}} \chi^{(k)}, \qquad \mathcal{J}_z^{\star(k)} = \partial_{\bar{y}} \chi^{(k)}. \tag{23}$$

If we define $(k+1)$th currents $\mathcal{J}_y^{\star(k+1)}$ and $\mathcal{J}_z^{\star(k+1)}$ as

$$\mathcal{J}_y^{\star(k+1)} \equiv \mathcal{D}_y \chi^{(k)} = \partial_y \chi^{(k)} - \mathcal{J}_y^\star \star \chi^{(k)}, \qquad \mathcal{J}_z^{\star(k+1)} \equiv \mathcal{D}_z \chi^{(k)} = \partial_z \chi^{(k)} - \mathcal{J}_z^\star \star \chi^{(k)}, \tag{24}$$

then the currents $\mathcal{J}_y^{\star(k+1)}$ and $\mathcal{J}_z^{\star(k+1)}$ are also conserved, i.e.

$$\begin{aligned}
\partial_{\bar{y}} \mathcal{J}_y^{\star(k+1)} + \partial_{\bar{z}} \mathcal{J}_z^{\star(k+1)} &= (\partial_{\bar{y}} \star \mathcal{D}_y + \partial_{\bar{z}} \star \mathcal{D}_z)\chi^{(k)}, \\
&= (\mathcal{D}_y \star \partial_{\bar{y}} + \mathcal{D}_z \star \partial_{\bar{z}})\chi^{(k)}, \text{ using equation (22)} \\
&= \mathcal{D}_y \star \partial_{\bar{y}} \chi^{(k)} - \mathcal{D}_z \star \partial_{\bar{z}} \chi^{(k)}, \\
&= \mathcal{D}_y \star \mathcal{D}_z \chi^{(k-1)} - \mathcal{D}_z \star \mathcal{D}_y \chi^{(k-1)} \\
&= [\mathcal{D}_y, \mathcal{D}_z]_\star \chi^{(k-1)} \\
&= 0.
\end{aligned}$$

In order to complete the induction, we set $\mathcal{J}_y^{\star(1)} = J^{\star -1} \star \partial_y J^\star$, $\mathcal{J}_z^{\star(1)} = J^{\star -1} \star \partial_z J^\star$ and $\chi^{(0)} = 1$. Note that the conservation of kth current implies the conservation of $(k+1)$th current and as a result an infinite number of conservation laws are obtained through induction.

We now relate the iterative procedure outlined above to the Lax pair (13) and (14) of the nc-(A)SDYM equations. From equations (23) and (24), we have

$$\partial_{\bar{z}} \chi^{(k)} = -\mathcal{D}_y \chi^{(k-1)}, \tag{25}$$

$$\partial_{\bar{y}} \chi^{(k)} = \mathcal{D}_z \chi^{(k-1)}. \tag{26}$$

Multiplying (25) and (26) by λ^{-k} and summing from $k = 1$ to $k = \infty$, we write

$$\sum_{k=1}^{\infty} \lambda^{-k} \partial_{\bar{z}} \chi^{(k)} = -\sum_{k=1}^{\infty} \lambda^{-k} \mathcal{D}_y \chi^{(k-1)}, \tag{27}$$

$$\sum_{k=1}^{\infty} \lambda^{-k} \partial_{\bar{y}} \chi^{(k)} = \sum_{k=1}^{\infty} \lambda^{-k} \mathcal{D}_z \chi^{(k-1)}. \tag{28}$$

As $\chi^{(0)} = 1$, the summation on the right-hand side of equations (27) and (28) can be extended to $k = 0$. By writing

$$\Psi = \sum_{k=0}^{\infty} \lambda^{-k} \chi^{(k)}, \tag{29}$$

equations (27) and (28) can be written as

$$(\partial_y + \lambda \partial_{\bar{z}}) \Psi(y, \bar{y}, z, \bar{z}; \lambda) = \mathcal{J}_y^\star \star \Psi(y, \bar{y}, z, \bar{z}; \lambda),$$

$$(\partial_z - \lambda \partial_{\bar{y}}) \Psi(y, \bar{y}, z, \bar{z}; \lambda) = \mathcal{J}_z^\star \star \Psi(y, \bar{y}, z, \bar{z}; \lambda).$$

This establishes the equivalence of the iterative procedure and the associated linear system of the nc-(A)SDYM equations. It is straightforward to derive a set of noncommutative version of compatible parametric Bäcklund transformation from the linear system (13), (14)

$$\mathcal{J}'^\star_y - \mathcal{J}^\star_y = \lambda^{-1} \partial_{\bar{z}} (J'^{\star -1} \star J^\star), \tag{30}$$

$$\mathcal{J}'^\star_z - \mathcal{J}^\star_z = -\lambda^{-1} \partial_{\bar{y}} (J'^{\star -1} \star J^\star), \tag{31}$$

along with the constraint

$$J^{\star -1} \star J'^\star - J'^{\star -1} \star J^\star = \lambda^{-1} I, \tag{32}$$

where $\mathcal{J}'^\star_y, \mathcal{J}'^\star_z$ satisfy the self-dual equation (15). The compatibility condition of (30), (31) implies that $\mathcal{J}^\star_y, \mathcal{J}'^\star_z, \mathcal{J}'^\star_y$ and $\mathcal{J}'^\star_z$ satisfy equation (15). This can be achieved by simply differentiating equation (30) with respect to $\bar{y}$ and equation (31) with respect to $\bar{z}$ and then adding both the results. For $J^\star$ and $J'^\star$ to be Hermitian, the Bäcklund transformation can be expressed in terms of two parameters with the constraints given by

$$\lambda^{-1} = \mathrm{e}^{\mathrm{i}\alpha}, \qquad J^{\star -1} \star J'^\star - J'^{\star -1} \star J^\star = \beta I, \tag{33}$$

where α and β are real constants. The two-parameter Bäcklund transformation (30)–(33) has been studied in [7] for the commutative case of $SU(n)$ SDYM equations. Also in the commutative case, the Bäcklund transformation of [7] reduces to the Pohlmeyer's Bäcklund transformation for the gauge group $SU(2)$ [6]. (For an earlier account of the Bäcklund transformation of $SU(2)$ Yang–Mills theory see [5].) The appearance of parameters and constraint in (30)–(33) is due to the correspondence to two-dimensional models such as the principal chiral model. A similar Bäcklund transformation has been constructed for a noncommutative principal chiral model in [56]. The constraints (33) appear while deriving equation (31) from equation (30). In fact, the derivation implies that the parameters α and β are real constants. For more details see the commutative case [7].

In the dimensional reduction of the nc-(A)SDYM equations to a noncommutative principal chiral model, the action of covariant derivatives on χ is defined as [56]

$$\mathcal{D}_0 \chi = \partial_0 \chi - \mathcal{J}_0^\star \star \chi, \qquad \mathcal{D}_1 \chi = \partial_1 \chi - \mathcal{J}_1^\star \star \chi,$$

where $\mathcal{J}_0^\star$ and $\mathcal{J}_1^\star$ are the components of the conserved currents of nc-principal chiral model. With these covariant derivatives the zero-curvature condition of the model can be expressed as

$$[\mathcal{D}_0, \mathcal{D}_1]_\star = 0.$$

The iteration procedure of the Brézin *et al* [55] generates an infinite sequence of non-local conserved quantities. The first two conserved quantities of the sequence are given by

$$\begin{aligned} \mathcal{Q}^{\star(1)} &= -\int_{-\infty}^{\infty} \mathrm{d}y \mathcal{J}_0^\star, \\ \mathcal{Q}^{\star(2)} &= -\int_{-\infty}^{\infty} \mathrm{d}y \mathcal{J}_1^\star + \int_{-\infty}^{\infty} \mathrm{d}y \mathcal{J}_0^\star \star \int_{-\infty}^{y} \mathrm{d}z \mathcal{J}_0^\star. \end{aligned} \tag{34}$$

It can be seen from the expression of the second conserved quantity that the density involves integral of the fields.

4. Binary Darboux transformation of nc-(A)SDYM equations

The Lax pair of the (A)SDYM equations can be used to construct binary Darboux transformation of the system as explained in [14] and the same procedure can be generalized to the case of nc-(A)SDYM equations that will lead to the construction of solutions of nc-(A)SDYM equations. Following [14] we proceed by rewriting the Lax pair (direct Lax pair) (13), (14) as

$$\begin{aligned} \partial_y \Psi &= -\lambda \partial_{\bar{z}} \Psi + \mathcal{J}_y^\star \star \Psi \\ \partial_z \Psi &= \lambda \partial_{\bar{y}} \Psi + \mathcal{J}_z^\star \star \Psi. \end{aligned} \tag{35}$$

Parallel to this linear system we define another Lax pair (dual Lax pair) for another matrix field Φ with spectral parameter λ' as

$$\begin{aligned} \partial_y \Phi &= -\lambda' \partial_{\bar{z}} \Phi - \Phi \star \mathcal{J}_y^\star \\ \partial_z \Phi &= \lambda' \partial_{\bar{y}} \Phi - \Phi \star \mathcal{J}_z^\star. \end{aligned} \tag{36}$$

The matrix solutions Ψ and Φ of the direct Lax pair (35) and dual Lax pair (36), respectively, are related to each other by

$$\Phi(y, \bar{y}, z, \bar{z}; \lambda) = A(\lambda y - \bar{z}, \lambda z + \bar{y}; \lambda) \star \Psi^{-1}(y, \bar{y}, z, \bar{z}; \lambda), \tag{37}$$

where $A(\lambda y + \bar{z}, \lambda z - \bar{y}; \lambda)$ is some arbitrary matrix function. Condition (37) is obtained by calculating $\partial_y (\Phi \star \Psi)$ and $\partial_z (\Phi \star \Psi)$ and solving the resulting equations for $\lambda' = \lambda$. The matrix $J^\star$ can be clearly related to the solution Φ of (36) and the matrix function A as

$$J^\star(y, \bar{y}, z, \bar{z}) \star A(-\bar{z}, \bar{y}) = \Psi(y, \bar{y}, z, \bar{z}; \lambda)|_{\lambda=0}. \tag{38}$$

Let ψ be a column solution and ϕ be a row solution of the Lax pair (35), (36) with spectral parameters μ and ν, respectively ($\mu \neq \nu$). If $\Psi[1]$ and $\Phi[1]$ are new matrix solutions satisfying the direct and dual Lax pairs (35) and (36), respectively, then through a projection operator (or projector) the one-fold Darboux transformation relates the matrix functions $\Psi[1]$ and $\Phi[1]$ with Ψ and Φ, respectively, by

$$\begin{aligned} \Psi[1] &= \left(I - \frac{\mu - \nu}{\lambda - \nu} P \right) \star \Psi, \\ \Phi[1] &= \Phi \star \left(I - \frac{\mu - \nu}{\mu - \lambda'} P \right), \end{aligned} \tag{39}$$

where P is the Hermitian projector, i.e. $P^\dagger = P = P^2$ (for the reality condition (10) to be satisfied). The corresponding solution $J^\star[1]$ of the nc-(A)SDYM equation (15) with a given known solution $J^\star$ is given by

$$J^\star[1] = \left(I + \frac{\mu - \nu}{\nu} P\right) \star J^\star, \tag{40}$$

where $J^\star$ is the solution of the Lax pair (35) at $\lambda = 0$. The solution (40) is interpreted as the new solution of the nc-(A)SDYM equations (15). The Lax pair for the matrix field $\Psi[1]$ is

$$\begin{aligned} \partial_y \Psi[1] &= -\lambda \partial_{\bar{z}} \Psi[1] + \mathcal{J}_y^\star[1] \star \Psi[1], \\ \partial_z \Psi[1] &= \lambda \partial_{\bar{y}} \Psi[1] + \mathcal{J}_z^\star[1] \star \Psi[1], \end{aligned} \tag{41}$$

and the dual Lax pair for is $\Phi[1]$

$$\begin{aligned} \partial_y \Phi[1] &= -\lambda' \partial_{\bar{z}} \Phi[1] - \Phi[1] \star \mathcal{J}_y^\star[1], \\ \partial_z \Phi[1] &= \lambda' \partial_{\bar{y}} \Phi[1] - \Phi[1] \star \mathcal{J}_z^\star[1]. \end{aligned} \tag{42}$$

Equations (41) and (42) show the covariance of the direct Lax pair and the dual Lax pair under the Darboux transformation (39). The transformation of $\Psi[1]$ and $\Phi[1]$ implies the transformation of the currents $\mathcal{J}_y^\star[1]$ and $\mathcal{J}_z^\star[1]$ as

$$\begin{aligned} \mathcal{J}_y^\star[1] &= \mathcal{J}_y^\star - (\mu - \nu) \frac{\partial P}{\partial \bar{z}}, \\ \mathcal{J}_z^\star[1] &= \mathcal{J}_z^\star + (\mu - \nu) \frac{\partial P}{\partial \bar{y}}, \end{aligned} \tag{43}$$

and the projector P satisfies the following equations:

$$\begin{aligned} \frac{\partial P}{\partial y} &= -\frac{1}{2} (\mu + \nu) \frac{\partial P}{\partial \bar{z}} - \left[\frac{1}{2} (\mu - \nu) \frac{\partial P}{\partial \bar{z}} - \mathcal{J}_y^\star, P\right]_\star, \\ \frac{\partial P}{\partial z} &= \frac{1}{2} (\mu + \nu) \frac{\partial P}{\partial \bar{y}} + \left[\frac{1}{2} (\mu - \nu) \frac{\partial P}{\partial \bar{y}} + \mathcal{J}_z^\star, P\right]_\star. \end{aligned} \tag{44}$$

In terms of the vector solutions ψ and ϕ of the direct and dual Lax pairs (35), (36) with the spectral parameters μ and ν, respectively, the projector P satisfying equation (44) can be expressed as

$$P = \psi \star (\phi, \psi)^{-1} \star \phi. \tag{45}$$

Here, we have seen that the projector P is expressed in terms of the solutions of the Lax pair (35) and (36), so that the solution $J^\star[1]$ will be expressed in terms of the solutions of the direct and dual Lax pairs.

The successive iteration of binary Darboux transformation gives the solutions of the direct and the dual Lax pairs as

$$\begin{aligned} \Psi[N] &= \left(I - \frac{\mu^{(N)} - \nu^{(N)}}{\lambda - \nu^{(N)}} P[N]\right) \star \cdots \star \left(I - \frac{\mu^{(1)} - \nu^{(1)}}{\lambda - \nu^{(1)}} P[1]\right) \star \Psi, \\ \Phi[N] &= \Phi \star \left(I - \frac{\mu^{(1)} - \nu^{(1)}}{\mu^{(1)} - \lambda'} P[1]\right) \star \cdots \star \left(I - \frac{\mu^{(N)} - \nu^{(N)}}{\mu^{(N)} - \lambda'} P[N]\right), \end{aligned} \tag{46}$$

and $P[i]$ is given by

$$P[i] = \psi^{(i)}[i-1] \star (\phi^{(i)}[i-1], \psi^{(i)}[i-1])^{-1} \star \phi^{(i)}[i-1], \tag{47}$$

where $\psi^{(i)}[i-1]$ and $\phi^{(i)}[i-1]$ $(i = 2, 3, \ldots N)$ are row and column solutions of the direct and dual Lax pairs (35) and (36) with spectral parameters $\mu^{(i)}$ and $\nu^{(i)}$, respectively, and are given by

$$\begin{aligned}
\psi^{(i)}[i-1] &= \left(I - \frac{\mu^{(i-1)} - \nu^{(i-1)}}{\mu^{(i)} - \nu^{(i-1)}} P[i-1]\right) \star \cdots \star \left(I - \frac{\mu^{(1)} - \nu^{(1)}}{\mu^{(i)} - \nu^{(1)}} P[1]\right) \star \psi^{(i)}, \\
\phi^{(i)}[i-1] &= \phi^{(i)} \star \left(I - \frac{\mu^{(1)} - \nu^{(1)}}{\mu^{(1)} - \nu^{(i)}} P[1]\right) \star \cdots \star \left(I - \frac{\mu^{(i-1)} - \nu^{(i-1)}}{\mu^{(i-1)} - \nu^{(i)}} P[i-1]\right).
\end{aligned} \tag{48}$$

The solution $\Psi[N]$ of the Lax pair (35) is obtained by iteration to N-fold binary Darboux transformation and $\Psi[N]$ in equation (46) corresponds to N-soliton solution of the nc-(A)SDYM equations. The reality condition (10) is satisfied if $P[i]$ is Hermitian.

For the N-fold binary Darboux transformation the direct Lax pair will be

$$\begin{aligned}
\partial_y \Psi[N] &= -\lambda \partial_{\bar{z}} \Psi[N] + \mathcal{J}_y^{\star}[N] \star \Psi[N], \\
\partial_z \Psi[N] &= \lambda \partial_{\bar{y}} \Psi[N] + \mathcal{J}_z^{\star}[N] \star \Psi[N],
\end{aligned} \tag{49}$$

and the dual Lax pair will be

$$\begin{aligned}
\partial_y \Phi[N] &= -\lambda' \partial_{\bar{z}} \Phi[N] - \Phi[N] \star \mathcal{J}_y^{\star}[N], \\
\partial_z \Phi[N] &= \lambda' \partial_{\bar{y}} \Phi[N] - \Phi[N] \star \mathcal{J}_z^{\star}[N].
\end{aligned} \tag{50}$$

The new currents $\mathcal{J}_y^{\star}[N]$ and $\mathcal{J}_z^{\star}[N]$ as solutions of (15) are obtained as

$$\begin{aligned}
\mathcal{J}_y^{\star}[N] &= \mathcal{J}_y^{\star}[N-1] - (\mu^{(N-1)} - \nu^{(N-1)}) \frac{\partial P[N-1]}{\partial \bar{z}}, \\
\mathcal{J}_z^{\star}[N] &= \mathcal{J}_z^{\star}[N-1] + (\mu^{(N-1)} - \nu^{(N-1)}) \frac{\partial P[N-1]}{\partial \bar{y}},
\end{aligned} \tag{51}$$

where the projector $P[N]$ satisfies the following equations:

$$\begin{aligned}
\frac{\partial P[N]}{\partial y} &= -\frac{1}{2}(\mu^{(N)} + \nu^{(N)}) \frac{\partial P[N]}{\partial \bar{z}} - \left[\frac{1}{2}(\mu^{(N)} - \nu^{(N)}) \frac{\partial P[N]}{\partial \bar{z}} - \mathcal{J}_y^{\star}[N], P[N]\right]_{\star}, \\
\frac{\partial P[N]}{\partial z} &= \frac{1}{2}(\mu^{(N)} + \nu^{(N)}) \frac{\partial P[N]}{\partial \bar{y}} + \left[\frac{1}{2}(\mu^{(N)} - \nu^{(N)}) \frac{\partial P[N]}{\partial \bar{y}} + \mathcal{J}_z^{\star}[N], P[N]\right]_{\star}.
\end{aligned} \tag{52}$$

The Nth iterated formulae obtained above generate N-soliton solutions of nc-(A)SDYM equations. The solution $J^{\star}[N]$ is now expressed in terms of the projector $P[i]$ as

$$J^{\star}[N] = \left(I + \frac{\mu^{(N)} - \nu^{(N)}}{\nu^{(N)}} P[N]\right) \star \cdots \star \left(I + \frac{\mu^{(1)} - \nu^{(1)}}{\nu^{(1)}} P[1]\right) \star J^{\star}. \tag{53}$$

If we denote the vector solution of (35) at the spectral parameter λ by $\tilde{\psi}$ and the vector solution of (36) at the spectral parameter λ' by $\tilde{\phi}$ and use the fact that $\tilde{\psi}[N] = 0$ for $\lambda = \mu^{(i)}$ with $\tilde{\psi} = \psi^{(i)}$ and $\tilde{\phi}[N] = 0$ for $\lambda = \nu^{(i)}$ with $\tilde{\phi} = \phi^{(i)}$, then the multiplicative ansatz (46) can also be expressed as

$$\begin{aligned}
\Psi[N] &= \left(I - \sum_{i,k=1}^{N} \frac{\mu^{(k)} - \nu^{(i)}}{\lambda - \nu^{(k)}} \psi^{(i)} \star (\phi^{(i)}, \psi^{(k)})^{-1} \star \phi^{(k)}\right) \star \Psi, \\
\Phi[N] &= \Phi \star \left(I - \sum_{i,k=1}^{N} \frac{\mu^{(k)} - \nu^{(i)}}{\mu^{(i)} - \lambda'} \psi^{(i)} \star (\phi^{(i)}, \psi^{(k)})^{-1} \star \phi^{(k)}\right),
\end{aligned} \tag{54}$$

and the solution $J^{\star}[N]$ will be expressed as

$$J^{\star}[N] = \left(I + \sum_{i,k=1}^{N} \frac{\mu^{(k)} - \nu^{(i)}}{\nu^{(k)}} \psi^{(i)} \star (\phi^{(i)}, \psi^{(k)})^{-1} \star \phi^{(k)}\right) \star J^{\star}. \tag{55}$$

The solution $\Psi[N]$ expressed in additive form is subjected to the reality condition (10). For condition (10) to be satisfied the projector $P[i]$ has to satisfy certain algebraic equations and the solution of these equations is obtained by taking the projectors $P[i]$ to be Hermitian and mutually orthogonal, i.e.

$$P^{\dagger}[i] = P[i] = P^2[i], \qquad P[i]P[j] = 0, \qquad \text{for} \quad i \neq j.$$

Such solutions have been discussed for a noncommutative (2 + 1)-dimensional modified $U(n)$ sigma model as a reduction of nc-(A)SDYM equations [50–53].

The formalism of binary Darboux transformation for the solutions of nc-(A)SDYM equations can be compared with the dressing approach for such solutions where the nc-(A)SDYM equations are reduced to a noncommutative modified $U(n)$ sigma model in (2 + 1) dimensions (see, e.g., [49–53] and references therein) and it is observed that both methods give the same results. Using the standard method of Riemann–Hilbert problem with zeros, the solution of the Lax pair is expressed in a simple form as [49–53]

$$\Psi[N] = \left(I - \sum_{k=1}^{N} \frac{1}{\lambda - \nu^{(k)}} R[k] \right) \star \Psi, \tag{56}$$

where matrix $R[k]$ is matrix function independent of λ. By comparing this with our result (54), we note that

$$R[k] = \sum_{i=1}^{N} (\mu^{(k)} - \nu^{(i)}) \psi^{(i)} \star (\phi^{(i)}, \psi^{(k)})^{-1} \star \phi^{(k)}. \tag{57}$$

Let us make few comments on the solution generating method outlined here and the dressing method outlined in [50–53]. In the binary Darboux transformation we combine the Darboux transformations for the direct and dual Lax pairs and express the solutions in terms of projectors. The projectors are expressed in terms of the solutions of both the Lax pairs. In dressing method introduced by Zakharov and Shabat [57–59], the solution of a system is obtained by reducing the solution of the spectral problem to that of a Riemann–Hilbert problems with zeros that implies the possibility of introducing projectors that relate solutions of Riemann–Hilbert problem in a simple algebraic form by using the analytic properties of the functions involved. On the other hand, the binary Darboux transformation reproduces same results without much use of the analytic properties.

5. Conclusions

In conclusion, we have derived an infinite set of non-local conservation laws by using iterative method of Brézin *et al* [55] and noncommutative version of a parametric Bäcklund transformation for nc-(A)SDYM equations. The nc-(A)SDYM equations are obtained as compatibility condition of a Lax pair. The Lax pair is further used to derive a binary Darboux transformation that generates solutions of nc-(A)SDYM equations. We conclude that noncommutative generalization of (A)SDYM equations preserves many of the integrability structures that are present in commutative (A)SDYM equations. The work can be further extended to construct the explicit solutions of the system for a given gauge group. It has been recently shown that most noncommutative integrable equations in three and less dimensions can be obtained by nc-(A)SDYM equations by suitable reductions (see [20–24] and reference therein). The applications of binary Darboux transformation for nc-(A)SDYM equations can further be investigated by understanding the Darboux transformation of noncommutative integrable equations in (2 + 1) and (1 + 1) dimensions. The Darboux transformation of noncommutative principal chiral model in (1 + 1) dimension has been investigated in [56], the method can be applied to obtain solutions of the noncommutative modified $U(n)$ sigma model in (2 + 1) dimensions as a reduction of nc-(A)SDYM equations. Another direction to pursue is to investigate the reductions of noncommutative supersymmetric (A)SDYM equations to noncommutative supersymmetric integrable models in three and less dimensions (e.g. [60]). Similarly, the integrable properties such as existence of an infinite number of continuity-like equations, Lax pair, Bäcklund and Darboux transformation can be investigated for the noncommutative supersymmetric (A)SDYM equations.

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